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On the propagator of a scalar field in AdS $\times$ S and in its plane wave limit

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Abstract: We discuss the scalar propagator on generic $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ backgrounds. For the conformally flat situations and masses corresponding to Weyl invariant actions, the propagator is powerlike in the sum of the chordal distances with respect to AdS_{d+1} and $\mathrm{S}^{d'+1}$. In these cases we analyze its source structure. In all other cases the propagator depends on both chordal distances separately. There an explicit formula is found for certain special mass values. For pure AdS we show how the well known propagators in the Weyl invariant case can be expressed as linear combinations of simple powers of the chordal distance. For $\mathrm{AdS}_5 \times \mathrm{S}^5$ we relate our propagator to the expression in the plane wave limit and find a geometric interpretation of the variables occurring in the known explicit construction on the plane wave. As a byproduct of comparing different techniques, including the KK mode summation, a theorem for summing certain products of Legendre and Gegenbauer functions is derived.

Keywords: Field Theories in Higher Dimensions, Penrose limit and pp-wave background, AdS-CFT and dS-CFT Correspondence.

Contents

1. Introduction

The AdS/CFT correspondence [1] relates $\mathcal{N} = 4$ super Yang-Mills gauge theory in Minkowski space to type II B string theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$ with some RR background flux. In the supergravity approximation one handles the fields in a Kaluza-Klein mode expansion with respect to the S^5. For calculations on the supergravity side the propagators of the whole spectrum of fields in the AdS_5 background are an essential technical ingredient. The simplest case to start with is of course the well known scalar propagator [2, 3]. Explicit tests of the AdS/CFT correspondence, beyond the supergravity approximation, remain a difficult task, since the relevant string spectrum in general is not available. In a limit of this correspondence, proposed by Berenstein, Maldacena and Nastase (BMN limit) [4], the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background itself is transformed via a Penrose limit [5] to a certain plane wave background found by Blau, Figueroa-O'Farrill, Hull and Papadopoulos [6, 7, 8]. In this background, for brevity called 'the plane wave', string theory is exactly quantizable [9], and thus enables independent checks of the duality, including string effects. In this plane wave background the separation between the AdS_5 and the S^5 part breaks down, and one has to take the limit on full 10-dimensional $\mathrm{AdS}_5 \times \mathrm{S}^5$ objects.

One of the crucial unsolved questions in this setting concerns the issue of holography [10]–[14]. In the Penrose limiting process the old 4-dimensional conformal boundary is put beyond the new plane wave space, which by itself has a one-dimensional conformal boundary. In [15] we started to investigate this issue and found, for each point remaining in the final plane wave, a degeneration of the cone of boundary reaching null geodesics into a single direction. To continue this program beyond geometric properties, we now want to study the limiting process for field theoretical propagators.

The so called bulk-to-boundary propagator plays an essential role in the holographic description of the AdS/CFT correspondence and is therefore a quantity of particular interest. However, as follows from the results in [15], a reasonable Penrose limit cannot be taken due to the fact that one of its legs ends at the boundary and hence lies outside the region of convergence to the plane wave. The situation is different for the bulk-to-bulk propagator. One has the choice to let both legs end within the region that converges to the plane wave in the limit. Although this propagator first seems to be of minor importance for the realization of holography and even for the computation of correlation functions [16], it might be very useful for defining a bulk-to-boundary propagator in the plane wave. A hint that this could be a promising direction is, that in pure AdS_{d+1} the knowledge of the bulk-to-bulk propagator is sufficient for deriving the corresponding bulk-to-boundary propagator.

Due to the existence of such a relation it seems to be worthwhile to study also in AdS $\times$ S the the bulk-to-bulk propagator to get information about the bulk-boundary correspondence in the plane wave limit. For brevity when we talk about 'the propagator' in the following we always understand it as the bulk-to-bulk one.

The scalar propagator in the plane wave has been constructed in [17] by a direct approach leaving the issue of its derivation via a limiting process from $\mathrm{AdS}_5 \times \mathrm{S}^5$ as an open problem.

Motivated by the above given questions, and because it is an interesting problem in its own right, we will study in this paper the construction of the scalar propagator on $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ spaces with radii R_1 and R_2, respectively. Allowing for generic dimensions d and d' as well as generic curvature radii R_1 and R_2 is very helpful to understand the general mechanism for the construction of the propagator. Of course only some of these spacetimes are parts of consistent supergravity backgrounds.

In section 2 we will focus on the differential equation defining the scalar propagator. Within this section we will be able to find the propagator in conformally flat situations, i.e. $R_1 = R_2$ and for masses corresponding to Weyl invariant actions.

The next two sections mainly serve as a kind of interpretation of the results of section 2. Using Weyl invariance, we map patchwise to flat space in section 3 and globally to the Einstein Static Universe (ESU) in section 4. This includes a discussion of global aspects of the solutions, like their boundary conditions and δ-source structure.

With the hope to get the propagator for generic masses, in section 5 we study its KK mode sum. We will be able to perform the sum for a linear relation between the conformal dimension of the KK mode and the quantum number parameterizing the eigenvalue of the laplacian on the sphere. Beyond the cases treated in the previous sections this applies to

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certain additional mass values, but fails to solve the full generic problem. As a byproduct, the comparison with the result of section 2 yields a theorem on the summation of certain products of Gegenbauer and Legendre functions.

In section 6 we will discuss the plane wave limit of $\mathrm{AdS}_5 \times \mathrm{S}^5$ in brief. We will explicitly show that the massless propagator on the full spacetime indeed reduces to the expression of [17]. Furthermore, we will present the limit of the full differential equation which is fulfilled by the propagator of massive scalar fields given in [17]. Finally, our results will be summarized in section 7.

In appendix A a detailed derivation of the relation between the bulk-to-bulk and the bulk-to-boundary propagator in the case of pure euclidean AdS_{d+1} is presented. Appendix B summarizes some of the relevant formulae for hypergeometric functions and spherical harmonics we needed for the analysis. Furthermore we will sketch an independent proof of the theorem that has been extracted from sections 2 and 5. Appendix C contains a short review of the embedding of the plane wave in flat spacetime.

2. The differential equation for the propagator and its solution

2.1 The scalar propagator on $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$

The scalar propagator is defined as the solution of[1]

$$(\Box_z - M^2) G(z,z') = \frac{i}{\sqrt{-g}} \delta(z,z') \,, \qquad (2.1)$$

with suitable boundary conditions at infinity. $\Box_z$ denotes the d'Alembert operator on $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$, acting on the first argument of the propagator $G(z,z')$. In the following we denote the coordinates referring to the AdS_{d+1} factor by x and those referring to the $\mathrm{S}^{d'+1}$ factor by y, i.e. $z = (x,y)$. We first look for solutions at $z \neq z'$ and discuss the behaviour at $z = z'$ afterwards.

AdS_{d+1} and $\mathrm{S}^{d'+1}$ can be interpreted as embeddings respectively in $\mathbb{R}^{2,d}$ and in $\mathbb{R}^{d'+2}$ with the help of the constraints[2]

$$-X_0^2 - X_{d+1}^2 + \sum_{i=1}^{d} X_i^2 = -R_1^2 \,, \qquad \sum_{i=1}^{d'+2} Y_i^2 = R_2^2 \,, \qquad (2.2)$$

where $X = X(x)$, $Y = Y(y)$ depend on the coordinates x and y, respectively. We define the chordal distances on both spaces to be

$$u(x,x') = (X(x) - X(x'))^2 \,, \qquad v(y,y') = (Y(y) - Y(y'))^2 \,. \qquad (2.3)$$

The distances have to be computed with the corresponding flat metrics of the embedding spaces that can be read off from (2.2). The chordal distance u is a unique function of x and x' if one restricts oneself to the hyperboloid. On the universal covering it is continued

[1] This normalization is consistent with the definition in [3], because in a continuation to euclidean space the factor i in front of the δ-source becomes -1 as in (A.1).

[2] More precisely, AdS_{d+1} is the universal covering of the hyperboloid in $\mathbb{R}^{2,d}$.

as a periodic function. For later use we note that on the hyperboloid and on the sphere the antipodal points $\tilde{x}$ and $\tilde{y}$ to given points x and y are defined by changing the sign of the embedding coordinates X and Y respectively. From (2.3) one then finds with $\tilde{u} = u(x, \tilde{x}')$, $\tilde{v} = v(y, \tilde{y}')$

$$u + \tilde{u} = -4R_1^2\,, \qquad v + \tilde{v} = 4R_2^2\,. \tag{2.4}$$

Using the homogeneity and isotropy of both AdS_{d+1} and $\mathrm{S}^{d'+1}$ it is clear that the propagator can depend on z, z' only via the chordal distances $u(x, x')$ and $v(y, y')$. Strictly speaking this at first applies only if AdS_{d+1} is restricted to the hyperboloid. Up to subtleties due to time ordering (see the end of section 4) this remains true also on the universal covering. The d'Alembert operator then simplifies to

$$\begin{aligned}
\Box_z &= \Box_x + \Box_y\,,\\
\Box_x &= 2(d+1)\Big(1 + \frac{u}{2R_1^2}\Big)\frac{\partial}{\partial u} + \Big(\frac{u^2}{R_1^2} + 4u\Big)\frac{\partial^2}{\partial u^2}\,,\\
\Box_y &= 2(d'+1)\Big(1 - \frac{v}{2R_2^2}\Big)\frac{\partial}{\partial v} - \Big(\frac{v^2}{R_2^2} - 4v\Big)\frac{\partial^2}{\partial v^2}\,.
\end{aligned} \tag{2.5}$$

One can now ask for a solution of (2.1) at $z \neq z'$ that only depends on the total chordal distance $u + v$. Indeed, using (2.5), it is easy to derive that such a solution exists if and only if

$$R_1 = R_2 = R\,, \qquad M^2 = \frac{d'^2 - d^2}{4R^2}\,. \tag{2.6}$$

Furthermore, it is necessarily powerlike and given by

$$G(z, z') \propto (u + v)^{-\frac{d+d'}{2}}\,. \tag{2.7}$$

Extending this to $z = z'$ we find just the right power for the short distance singularity to generate the δ-function on the r.h.s. of (2.1). Hence after fixing the normalization we end up with

$$G(z, z') = \frac{\Gamma(\frac{d+d'}{2})}{4\pi^{\frac{d+d'}{2}+1}} \frac{1}{(u + v + i\varepsilon(t, t'))^{\frac{d+d'}{2}}}\,. \tag{2.8}$$

Note that due to (2.4) besides the singularity at $z = z'$ there is another one at the total antipodal point where $z = \tilde{z}' = (\tilde{x}', \tilde{y}')$. We have introduced an $i\varepsilon$-prescription by replacing $u \to u + i\varepsilon$, where ε depends explicitly on time. We will comment on this in section 4. In particular, we will see that on the universal covering of the hyperboloid the singularity at the total antipodal point does not lead to an additional δ-source on the r.h.s. of (2.1).

Scalar fields with mass m^2 in AdS_{d+1} via the AdS/CFT correspondence are related to CFT fields with conformal dimension

$$\Delta_\pm(d, m^2) = \frac{1}{2}\Big(d \pm \sqrt{d^2 + 4m^2R_1^2}\Big)\,. \tag{2.9}$$

Note that the exponent of $(u + v)$ in the denominator of the propagator (2.8) is just equal to $\Delta_+(d, M^2)$. From the AdS_{d+1} point of view the $(d + d' + 2)$-dimensional mass M^2 is the mass of the KK zero mode of the sphere. We will say more on these issues in section 5.

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For completeness let us add another observation. Disregarding for a moment the source structure, under the conditions (2.6) there is a solution of (2.1), that depends only on $(u-v)$. The explicit form is

$$\tilde{G}(z,z') \propto \frac{1}{(u-v+4R^2+i\varepsilon(t,t'))^{\frac{d+d'}{2}}} \,. \tag{2.10}$$

It has the same asymptotic falloff as (2.8). But due to (2.4) it has singularities only at the semi-antipodal points where $z = z'_{\rm s} = (x',\tilde{y}')$ and $z = \tilde{z}'_{\rm s} = (\tilde{x}',y')$. We will say more on $\tilde{G}(z,z')$ in sections 3 and 4.

At the end of this subsection we give a simple interpretation of the conditions (2.6). The equality of the radii is exactly the condition for conformal flatness of the complete product space $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ as a whole. Describing the AdS metric in Poincaré coordinates $(x = (x^0, x^1, \ldots, x^{d-1}, x_\perp))$ one finds

$$\mathrm{d}s^2 = \frac{R_1^2}{x_\perp^2}\big(- (\mathrm{d}x^0)^2 + \mathrm{d}x_\perp^2 + \mathrm{d}\vec{x}^2 \big) \,. \tag{2.11}$$

For $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ one thus obtains

$$\mathrm{d}s^2 = \frac{R_1^2}{x_\perp^2}\left(-(\mathrm{d}x^0)^2 + \mathrm{d}x_\perp^2 + \mathrm{d}\vec{x}^2 + \frac{R_2^2}{R_1^2} x_\perp^2 \, \mathrm{d}\Omega_{d'+1}^2 \right) , \tag{2.12}$$

which is obviously conformally flat if $R_1 = R_2$. That this is also necessary for conformal flatness follows from an analysis of the corresponding Weyl tensor. Furthermore, the mass condition just singles out the case of a scalar field coupled in Weyl invariant manner to the gravitational background. The corresponding D-dimensional action is

$$S = -\frac{1}{2}\int \mathrm{d}^D z \sqrt{-g}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{D-2}{4(D-1)}\mathcal{R}\phi^2 \right] . \tag{2.13}$$

Inserting the constant curvature scalar $\mathcal{R}$ for $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ with equal radii one gets for the mass just the value in (2.6).

Altogether in this subsection we have constructed the scalar $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ propagator for the case of Weyl invariant coupling to the metric in conformally flat situations. The Weyl invariant coupled field is the natural generalization of the massless field in flat space.

2.2 A remark on the propagator on pure AdS_{d+1}

Having found for $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ such a simple expression for the scalar propagator, one is wondering whether the well known AdS propagators can also be related to simple powers of the chordal distance.

The general massive scalar propagator on pure AdS_{d+1} space corresponding to the two distinct conformal dimensions $\Delta_\pm$ defined in (2.9) with generic mass values is given by [2, 3]

$$\begin{aligned} G_{\Delta_\pm}(x,x') &= \frac{\Gamma(\Delta_\pm)}{R_1^{d-1} 2\pi^{\frac{d}{2}}\Gamma(\Delta_\pm - \frac{d}{2}+1)}\left(\frac{\xi}{2}\right)^{\Delta_\pm} F\left(\frac{\Delta_\pm}{2}, \frac{\Delta_\pm}{2}+\frac{1}{2}; \Delta_\pm - \frac{d}{2}+1; \xi^2\right) , \\ \xi &= \frac{2R_1^2}{u+2R_1^2} \,. \end{aligned} \tag{2.14}$$

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Again here a powerlike solution of (2.1) (but now using only the AdS d'Alembert operator of (2.5) and replacing M^2 by the AdS mass m^2) exists for the Weyl invariant coupled mass value[3]

$$m^2 = \frac{1-d^2}{4R_1^2}\,. \tag{2.15}$$

The related value for the conformal dimension from (2.9) is then $\Delta_\pm = \frac{d\pm 1}{2}$. The powerlike solution is given by

$$G(x,x') = \frac{\Gamma(\frac{d-1}{2})}{4\pi^{\frac{d+1}{2}}} \frac{1}{(u+i\varepsilon(t,t'))^{\frac{d-1}{2}}}\,. \tag{2.16}$$

In contrast to the $\mathrm{AdS}_{d+1}\times \mathrm{S}^{d'+1}$ case here the exponent of u is given by $\Delta_-(d,m^2)$. We have again kept the option of a time dependent $i\varepsilon(t,t')$ and will comment on it in section 4.

The above solution can indeed be obtained from (2.14) by taking the sum of the expressions for Δ_+ and Δ_-. In addition one finds another simple structure by taking the difference. They are given by

$$\begin{aligned} \frac{1}{2}(G_{\Delta_-}+G_{\Delta_+}) &= \frac{\Gamma(\frac{d-1}{2})}{4\pi^{\frac{d+1}{2}}} \frac{1}{(u+i\varepsilon(t,t'))^{\frac{d-1}{2}}}\,, \\ \frac{1}{2}(G_{\Delta_-}-G_{\Delta_+}) &= \frac{\Gamma(\frac{d-1}{2})}{4\pi^{\frac{d+1}{2}}} \frac{1}{(u+4R^2+i\varepsilon(t,t'))^{\frac{d-1}{2}}}\,. \end{aligned} \tag{2.17}$$

Both expressions are derived by using (2.14) and (B.1) of appendix B. The first combination has the right short distance singularity to be a solution of (2.1). The second combination resembles (2.10). We will say more on this in sections 3 and 4.

2.3 Comment on masses and conformal dimensions on AdS_{d+1}

On AdS spaces one has to respect the Breitenlohner-Freedman bounds [18, 19]. To get real values for $\Delta_\pm$ requires

$$m^2 \geq -\frac{d^2}{4R_1^2}\,. \tag{2.18}$$

Furthermore, the so called unitarity bound requires

$$\Delta > \frac{d-2}{2}\,. \tag{2.19}$$

This implies that for $-\frac{d^2}{4R_1^2} \leq m^2 < \frac{4-d^2}{4R_1^2}$ both, Δ_+ and Δ_- are allowed. On the other side for $\frac{4-d^2}{4R_1^2} \leq m^2$ only Δ_+ is allowed.

The masses for Weyl invariant coupling are $\frac{1-d^2}{4R_1^2}$ and $\frac{d'^2-d^2}{4R_1^2}$ for AdS_{d+1} and $\mathrm{AdS}_{d+1}\times \mathrm{S}^{d'+1}$, respectively. Hence in our Weyl invariant cases for pure AdS Δ_+ and Δ_- are allowed while for $\mathrm{AdS}_{d+1}\times \mathrm{S}^{d'+1}$ with $d'>1$ only Δ_+ is allowed.

[3] Pure AdS spaces are conformally flat.

3. Derivation of the propagator from the flat space one

In the previous section we have shown that a simple powerlike solution of (2.1) can be found if the underlying spacetime is AdS_{d+1} or a conformally flat product space $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ and if the corresponding scalar field is Weyl invariant coupled to the curvature of the background. Both properties allow for a mapping of the differential equation, the scalar field and the propagator to flat space. The other way around, one can use Weyl invariance in this special case to construct the propagator of Weyl invariant coupled fields on conformally flat backgrounds from the flat space massless propagator.

We will use this standard construction to rederive the $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ expressions (2.8) and (2.10) from the flat space solutions.

The relevant Weyl transformation in a D-dimensional manifold is

$$g_{\mu\nu} \to \varrho\, g_{\mu\nu}\,, \qquad \phi \to \phi' = \varrho^{\frac{2-D}{4}}\phi\,. \tag{3.1}$$

If then the metric is of the form $g_{\mu\nu}(z) = \varrho(z)\,\eta_{\mu\nu}$ one finds the following relation between the propagator in curved and flat space

$$G(z,z') = \big(\varrho(z)\,\varrho(z')\big)^{\frac{2-D}{4}} G_{\text{flat}}(z,z')\,, \qquad G_{\text{flat}}(z,z') = \frac{\Gamma(\frac{D-2}{2})}{4\pi^{\frac{D}{2}}} \frac{1}{((z-z')^2+i\epsilon)^{\frac{D-2}{2}}}\,. \tag{3.2}$$

It can be derived either by formal manipulations with the corresponding functional integral or by using the covariance properties of the defining differential equation.[4]

Applying the formula first to pure AdS one gets in Poincaré coordinates (2.11)

$$G(x,x') = \frac{\Gamma(\frac{d-1}{2})}{R_1^{d-1} 4\pi^{\frac{d+1}{2}}} \left(\frac{1}{x_\perp x'_\perp}\left[(x_\perp - x'_\perp)^2 - (x^0 - x'^0)^2 + (\vec{x}-\vec{x}')^2 + i\epsilon\right]\right)^{\frac{1-d}{2}}. \tag{3.3}$$

Using the relation between Poincaré coordinates and the coordinates in the embedding space, see e.g. [20], it is straightforward to verify that this with (2.3) is equal to (2.16).

The Poincaré patch of pure AdS_{d+1} is conformal to a flat half space with $x_\perp \geq 0$. $x_\perp = 0$ corresponds to the conformal boundary of AdS. Let us first disregard that the flat half space represents only one half of AdS_{d+1} and discuss global issues later. We can then implement either Dirichlet or Neumann boundary conditions by the standard mirror charge method. To $x = (x_\perp, x^0, x^1, \dots, x^{d-1})$ we relate the mirror point[5]

$$\tilde{x} = (-x_\perp, x^0, x^1, \dots, x^{d-1}) \tag{3.4}$$

and the mirror propagator by

$$\tilde{G}_{\text{flat}}(x,x') = G_{\text{flat}}(x,\tilde{x}')\,. \tag{3.5}$$

Then $\frac{1}{2}(G_{\Delta_-} - G_{\Delta_+})$ in the second line of (2.17) turns out to be just the Weyl transformed version of $\tilde{G}_{\text{flat}}(x,x')$. Equivalently we can state, that G_{Δ_+} and G_{Δ_-} are the Weyl transformed versions respectively of the Dirichlet and Neumann propagator in the flat halfspace.

[4] Of course, the discussion has to be completed by considering also the boundary conditions.

[5] Using $x_\perp < 0$ for parameterizing the second Poincaré patch the mirror point is at the antipodal position on the hyperboloid.

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The situation is different for $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ spacetimes. According to (2.12), $x_\perp \geq 0$ becomes a radial coordinate of a full $(d'+2)$-dimensional flat subspace of a total space with coordinates

$$z = \big(x_0, \vec{x}, x_\perp \frac{\vec{Y}}{R}\big) \,, \tag{3.6}$$

where $\vec{Y}^2 = R^2$ are the embedding coordinates of $S^{d'+1}$. The boundary of the AdS part is mapped to the origin of the $(d'+2)$-dimensional subspace. Similarly to the pure AdS case, $G(z,z')$ from (2.8) is the Weyl transform of $G_{\text{flat}}(z,z')$. To see this one has to cast the length square on the $(d'+2)$-dimensional subspace, which appears in the denominator of the propagator, into the form

$$\frac{1}{R^2}(x_\perp \vec{Y} - x'_\perp \vec{Y}')^2 = x_\perp^2 + x_\perp'^2 - 2\frac{x_\perp x'_\perp}{R^2} \vec{Y}\vec{Y}' = (x_\perp - x'_\perp)^2 + \frac{x_\perp x'_\perp}{R^2} v \,. \tag{3.7}$$

In addition, with

$$z_{\mathrm{s}} = \left(x_0, \vec{x}, -x_\perp \frac{\vec{Y}}{R}\right) , \qquad \tilde{G}_{\text{flat}}(z,z') = G_{\text{flat}}(z,z'_{\mathrm{s}}) \tag{3.8}$$

we find that the second simple solution (2.10) is the Weyl transformed version of $\tilde{G}_{\text{flat}}(z,z')$.

The coordinates (3.6) and (3.8) are related by replacing $\vec{Y}$ by $-\vec{Y}$, i.e. z_{s} is related to z by going to the antipodal point in the sphere, according to the definition of z_{s} after (2.10). The two points z, z_{s} are elements of $\mathbb{R}^{d+d'+2}$ lying in the first Poincaré patch where $x_\perp \geq 0$.

As we mentioned before, one has to be careful with global issues. We work in the Poincaré patch that only covers points with $x_\perp \geq 0$. It is easy to see that the coordinates (3.6) of z and (3.8) of z_{s} remain unchanged if one simultaneously replaces $x_\perp$ by $-x_\perp$ and $\vec{Y}$ by $-\vec{Y}$. This operation switches from z and z_{s} respectively to the total antipodal positions $\tilde{z}$ and $\tilde{z}_{\mathrm{s}}$, that are covered by a second Poincaré patch with $x_\perp < 0$. Thus the latter points, being elements of the complete manifold, are not covered by the first Poincaré patch. In the context of pure AdS_{d+1}, the mirror point $\tilde{x}$ in (3.4) related to x is outside of the first Poincaré patch but it is still a point in AdS_{d+1} covered by the second Poincaré patch. Hence $\tilde{x}$ is not an element of the flat half space that is conformal to the first Poincaré patch. We will now analyze the global issues more carefully by working with the corresponding ESU.

4. Relation to the ESU

AdS_{d+1} and $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ with $R_1 = R_2$ are conformal to respectively one half and to the full ESU of the corresponding dimension. This can be easily seen in a certain set of global coordinates where the metric of AdS_{d+1} assumes the form

$$\mathrm{d}s^2_{\mathrm{AdS}} = R_1^2 \sec^2 \bar{\rho} \big(- \mathrm{d}t^2 + \mathrm{d}\bar{\rho}^2 + \sin^2 \bar{\rho}\, \mathrm{d}\Omega^2_{d-1} \big) \,, \tag{4.1}$$

where $0 \leq \bar{\rho} < \pi/2$. The corresponding ESU has the topology $\mathbb{R} \times \mathrm{S}^d$ and its metric is given by the expression in parentheses. The conformal map between AdS_{d+1} and ESU_{d+1} has been used in [21] at $d = 3$ to find consistent quantization schemes on AdS_4. In case of the Weyl invariant mass value (2.15) the quantization prescription on the ESU leads to two different descriptions for pure AdS. One can either choose transparent boundary conditions or reflective boundary conditions at the image of the AdS boundary. The reflectivity of the boundary is guaranteed for either Dirichlet or Neumann boundary conditions. This is realized by choosing a subset of modes with definite symmetry properties, whereas in the transparent case all modes are used. Quantization in the reflective case leads one to the solutions $G_{\Delta_\pm}$. These results motivate why we will work on the ESU in the following. We will find the antipodal points and see how the mirror charge construction works. Then we will discuss what this implies for the well known propagators in AdS_{d+1} and our solutions for $\mathrm{AdS}_{d+1}\times \mathrm{S}^{d'+1}$ in the Weyl invariant cases. In the coordinates (4.1) a point $\tilde{x}$ antipodal to the point $x=(t,\bar{\rho},x_\Omega)$ in AdS_{d+1} is given by

$$\tilde{x}=(t+\pi,\bar{\rho},\tilde{x}_\Omega)\,, \tag{4.2}$$

where x_Ω denotes the angles of the $(d-1)$-dimensional subsphere of AdS_{d+1} with embedding coordinates ω_i, which then obey

$$\omega_i(\tilde{x}_\Omega)=-\omega_i(x_\Omega)\,. \tag{4.3}$$

The above relation (4.2) must not be confused with the relation between two points that are antipodal to each other on the sphere of the ESU at fixed time.

We now want to visualize the above relation on the sphere of the ESU. For convenience we choose AdS_2 such that the ESU has topology $\mathbb{R}\times \mathrm{S}^1$. The subsphere of AdS_2 is given by $\mathrm{S}^0=\{-1,1\}$ such that we have $\omega=\pm1$. Hence, the transformation of x_Ω as prescribed in (4.2) becomes a flip between the two points of the S^0. The information contained in S^0 can be traded for an additional sign information of $\bar{\rho}$, and therefore the transformation from x_Ω to $\tilde{x}_\Omega$ simply corresponds to an reflection at $\bar{\rho}=0$. We will now describe the time shift. After the transformation of the spatial coordinates is performed, one has found the antipodal event at time $t+\pi$. To relate it to an event at the original time t one simply travels back in time along any null geodesics that crosses the spatial position of the antipodal event. On the ESU these null geodesics are clearly great circles. They meet at two points on the sphere. One is at the spatial position of the event and the other point is the antipodal point on the sphere of the ESU. The time it takes for a massless particle to travel between these two points is given by π, see figure 1.

In this way one now arrives at an event that can have caused the event at later time $t+\pi$, and that has the same time coordinate as x, and its coordinate value $\bar{\rho}$ is given by a reflection at $\bar{\rho}=\pi/2$ on S^1. As $\bar{\rho}=\pi/2$ is the position of the AdS boundary, the mirror image to x is situated outside of the region that corresponds to AdS. The effect of the original source at x in combination with the mirror source either at $\tilde{x}$ as given in (4.2) or at equal times mirrored at the boundary is that a light ray that travels to the boundary of AdS is reflected back into the interior.

Let us now discuss what happens in the case of $\mathrm{AdS}_{d+1}\times \mathrm{S}^{d'+1}$. The point $z=(t,\bar{\rho},x_\Omega,y)$ possesses the total antipodal point $\tilde{z}$ and the two semi-antipodal points z_{s} and $\tilde{z}_{\mathrm{s}}$ given by

$$\tilde{z}=(t+\pi,\bar{\rho},\tilde{x}_\Omega,\tilde{y})\,,\qquad z_{\mathrm{s}}=(t,\bar{\rho},x_\Omega,\tilde{y})\,,\qquad \tilde{z}_{\mathrm{s}}=(t+\pi,\bar{\rho},\tilde{x}_\Omega,y)\,, \tag{4.4}$$

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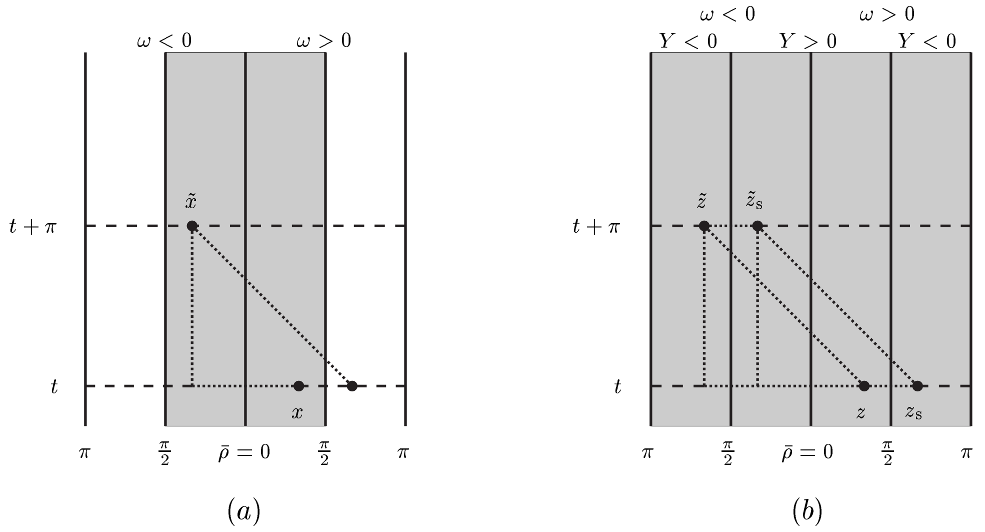

Figure 1: AdS_2 (a) and $\mathrm{AdS}_2\times \mathrm{S}^0$ (b) conformally mapped to the corresponding ESU. The regions that are covered are displayed as gray-filled regions. The ESU is given by a cylinder such that one has to identify the two boundaries of the strip where $\rho=\pi$. The two points of the S^0 within AdS_2 and of the extra factor S^0 in the product space are $\omega=\pm 1$ and $\frac{Y}{R}=\pm 1$, respectively. $\tilde{x}$ and $\tilde{z}$, $z_{\rm s}$, $\tilde{z}_{\rm s}$ are the antipodal points to x and z in respectively AdS_2 and $\mathrm{AdS}_2\times\mathrm{S}^0$. They are constructed by following the lines with small dashsize. The horizontal direction corresponds to the transformation in the space coordinates and the vertical one is associated to the time shift. The diagonal lines then point to the source at the corresponding conjugate point where null geodesics intersect. The conjugate points can be regarded as effective time shifted sources with the same time coordinate as the original event x or z.

where x_Ω is as in the pure AdS_{d+1} case and fulfills (4.3) and y are all angle coordinates of $S^{d'+1}$.

In figure 1 the case of $\mathrm{AdS}_2\times\mathrm{S}^0$, is shown. The effect of the factor S^0 can be alternatively described by adding to the range $0\le\bar{\rho}\le\pi/2$ the interval $\pi/2\le\bar{\rho}\le\pi$. This is possible because in the ESU at $\bar{\rho}=\pi/2$ the S^0 shrinks to a point. The complete ESU is now covered by the image of $\mathrm{AdS}_2\times\mathrm{S}^0$. The map to an antipodal position within the AdS_2 factor is as before, one finds the spatial coordinates by reflecting at $\bar{\rho}=0$. Within the S^0 factor, the antipodal position is found by reflecting at $\bar{\rho}=\pi/2$. Using this, it can be seen that w.r.t. the point z, the point $\tilde{z}$ is at the antipodal position on the S^1 of the ESU. Traveling back in time from $t+\pi$ to t along a null geodesic, one arrives at z from where one started. In the same way, the two semi-antipodal points $z_{\rm s}$, $\tilde{z}_{\rm s}$ are connected with each other by light rays. On the sphere of the ESU the z and $z_{\rm s}$ are related by a reflection at $\bar{\rho}=\pi/2$. Here, in contrast to the case of AdS_2, even the mirror events at equal times are situated within the image of $\mathrm{AdS}_2\times\mathrm{S}^0$. The above results are straightforwardly generalized to arbitrary dimensions.

Coming back to the discussion in section 3, we can now make more precise statements about the mirror charge method to impose definite boundary conditions at $\bar{\rho}=\frac{\pi}{2}$. A linear combination of the two solutions like in (2.17) does not necessarily generate additional δ-sources on the r.h.s. of the differential equation (2.1), although both powerlike solutions in (2.17) have singularities within AdS_{d+1}, the expression in the first line has one at $x = x'$ and the expression in the second line has one at $x = \tilde{x}'$. The singularity of the second expression only appears at $t = t' + \pi$, and its contribution to the r.h.s. of the differential equation (2.1) depends on the time ordering prescription. In the cases where the θ-function used for time ordering has an additional step at $t = t' + \pi$, a second δ-function is generated (see [21] for a discussion of AdS_4). With the standard time ordering one finds that $G_{\Delta_\pm}$ are solutions with a source at $x = x'$ only. For AdS_4 this was obtained in [22].

The situation is different for $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$, where the propagator (2.8) has singularities at $z = z'$, $z = \tilde{z}'$ and the second solution (2.10) has singularities at $z = z'_{\mathrm{s}}$, $z = \tilde{z}'_{\mathrm{s}}$. Again, whether the singularities at $z = \tilde{z}'$ and $z = \tilde{z}'_{\mathrm{s}}$ appear as δ-sources on the r.h.s. of the differential equation (2.1), depends on the chosen time ordering. However in contrast to the pure AdS_{d+1} case, the singularity of the second solution (2.10) at $z = z'_{\mathrm{s}}$ always leads to a δ-source on the r.h.s. of (2.1) but at the wrong position. This result corresponds to the above observation on the ESU that the mirror sources at equal times are not part of the image of AdS_{d+1} but of $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$.

At the end let us give some comments on the $i\varepsilon(t,t')$-prescription. First of all, one has to introduce it in all expressions (2.8), (2.10) and (2.17), since all of them have singularities at coincident or antipodal positions. Secondly, as worked out for AdS_4, a time independent $\varepsilon(t,t') = \epsilon$ refers to taking the step function $\theta(\sin(t-t'))$ for time ordering [21] which is appropriate if one restricts oneself to the hyperboloid. Standard time ordering with $\theta(t-t')$, being appropriate on the universal covering, yields a time dependent $\varepsilon(t,t') = \epsilon\,\mathrm{sgn}((t-t')\sin(t-t'))$ [22]. As mentioned in section 2, due to the time dependence of $\varepsilon(t,t')$, the coordinate dependence of the solutions is not entirely included in u and v.

5. Mode summation on $\mathbf{AdS}_{d+1} \times \mathbf{S}^{d'+1}$

In this section we will use the propagator on pure AdS_{d+1} given by (2.14) and the spherical harmonics on $\mathrm{S}^{d'+1}$ to construct the propagator on $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ via its mode expansion, summing up all the KK modes. We will be able to perform the sum only for special mass values where the conformal dimensions $\Delta_\pm$ of the scalar modes are linear functions of l, with l denoting the lth mode in the KK tower. Even a mixing of several scalar modes of this kind is allowed. The mixing case is interesting because it occurs in supergravity theories on $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d+1}$ backgrounds [23]–[27]. For example in type II B supergravity in $\mathrm{AdS}_5 \times \mathrm{S}^5$ the mass eigenstates of the mixing matrix for scalar modes [26, 27] correspond to the bosonic chiral primary and descendant operators in the AdS/CFT dictionary [28]. For these modes $\Delta_\pm$ depend linearly on l.

The main motivation for investigating the mode summation was the hope to find the propagator for generic mass values. But forced to stay in a regime of a linear $\Delta_\pm$ versus l relation we can give up the condition of conformal flatness, but remain restricted to special mass values. We nevertheless present this study since several interesting aspects are found along the way. Furthermore, in the literature it is believed that an explicit computation of the KK mode summation is too cumbersome [17]. We will show how to deal with the mode summation by discussing the $\mathrm{AdS}_3 \times \mathrm{S}^3$ case first, allowing for unequal radii but

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necessarily a special mass value. The result will then be compared to the expressions in the previous sections by specializing to equal embedding radii.

Having discussed this special case we will comment on the modifications which are necessary to deal with generic $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ spacetimes.

The results of the previous sections in connection with the expression for the mode summation in the conformally flat and Weyl invariant coupled case lead to the formulation of a summation rule for a product of Legendre functions and Gegenbauer polynomials. An independent proof of this rule is given in appendix B. With this it is possible to discuss the results in generic dimensions without doing all the computations explicitly. Furthermore, the sum rule might be useful for other applications, too.

For the solution of (2.1) we make the following ansatz[6]

$$G(z,z') = \frac{1}{R_2^{d'+1}} \sum_I G_I(x,x') Y^I(y) Y^{*I}(y')\,, \tag{5.1}$$

where we sum over the multiindex $I = (l, m_1, \dots, m_{d'})$ such that $l \geq m_1 \geq \cdots \geq m_{d'-1} \geq |m_{d'}| \geq 0$, Y^I denote the spherical harmonics on $\mathrm{S}^{d'+1}$, and '$*$' means complex conjugation. Some useful relations for the spherical harmonics can be found in appendix B.

The mode dependent Green function on AdS_{d+1} then fulfills

$$\left(\Box_x - M^2 - \frac{l(l+d')}{R_2^2}\right) G_I(x,x') = \frac{i}{\sqrt{-g_{\mathrm{AdS}}}} \delta(x,x')\,, \tag{5.2}$$

which follows when decomposing the d'Alembert operator like in (2.5) and using (B.6). The solution of this equation was already given in (2.14), into which the (now KK mode dependent) conformal dimensions enter. They were already defined in (2.9), and the AdS mass is a function of the mode label l

$$m^2 = M^2 + m_{\mathrm{KK}}^2 = M^2 + \frac{l(l+d')}{R_2^2}\,. \tag{5.3}$$

In the following as a simple example we will present the derivation of the propagator on $\mathrm{AdS}_3 \times \mathrm{S}^3$ via the KK mode summation. Compared to the physically more interesting $\mathrm{AdS}_5 \times \mathrm{S}^5$ background the expressions are easier and the general formalism becomes clear.

Evaluating (2.14) for $d = d' = 2$ the AdS_3 propagator for the lth KK mode is given by

$$G_\Delta(x,x') = \frac{1}{R_1 2^{\Delta+1}\pi} \xi^\Delta F\left(\frac{\Delta}{2}, \frac{\Delta}{2} + \frac{1}{2}; \Delta; \xi^2\right) = \frac{1}{R_1 4\pi} \frac{1+\sqrt{1-\xi^2}}{\sqrt{1-\xi^2}} \left[\frac{\xi}{1+\sqrt{1-\xi^2}}\right]^\Delta\,. \tag{5.4}$$

From (B.3), (2.9) and (5.3) one finds that the mode dependent positive branch of the conformal dimension reads

$$\Delta = \Delta_+ = 1 + \frac{R_1}{R_2}\sqrt{\frac{R_2^2}{R_1^2} + l(l+2) + M^2 R_2^2}\,. \tag{5.5}$$

[6]This ansatz is designed to generate a solution that corresponds to (2.8). If one wants to generate a solution corresponding to (2.10) one has to replace either y or y' by the corresponding antipodal coordinates $\tilde{y}$ or $\tilde{y}'$.

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The spherical part follows from (B.7) of appendix B where we discuss it in more detail and is given by

$$\sum_{m_1 \geq |m_2| \geq 0}^{l} Y^I(y) Y^{*I}(y') = \frac{(l+1)}{2\pi^2} C_l^{(1)}(\cos\Theta)\,, \qquad \cos\Theta = \frac{Y \cdot Y'}{R_2^2} = 1 - \frac{v}{2R_2^2}\,. \tag{5.6}$$

Remember that the $C_l^{(\beta)}$ denote the Gegenbauer polynomials and Y, Y' in the formula for Θ are the embedding space coordinates of the sphere, compare with (2.2) and (2.3). One thus obtains from (5.1)

$$G(z,z') = \frac{1}{8\pi^3 R_1 R_2^3} \frac{1+\sqrt{1-\xi^2}}{\sqrt{1-\xi^2}} \sum_{l=0}^{\infty} (l+1) \Big[\frac{\xi}{1+\sqrt{1-\xi^2}}\Big]^{\Delta} C_l^{(1)}(\cos\Theta)\,. \tag{5.7}$$

In this formula Δ is a function of the mode parameter l and we can explicitly perform the sum only for special conformal dimensions which are linear functions of l

$$\Delta = \Delta_+ = \frac{R_1}{R_2} l + \frac{R_1 + R_2}{R_2}\,, \tag{5.8}$$

following from (5.5) after choosing the special mass value

$$M^2 = \frac{1}{R_2^2} - \frac{1}{R_1^2}\,. \tag{5.9}$$

The sum then simplifies and can explicitly be evaluated by a reformulation of the l-dependent prefactor as a derivative and by using (B.8)

$$\sum_{l=0}^{\infty} (l+1) q^l C_l^{(1)}(\eta) = \left(q\frac{\partial}{\partial q} + 1\right) \sum_{l=0}^{\infty} q^l C_l^{(1)}(\eta) = \frac{1-q^2}{(1-2q\eta+q^2)^2}\,. \tag{5.10}$$

With the replacements

$$q = \Big[\frac{\xi}{1+\sqrt{1-\xi^2}}\Big]^{\frac{R_1}{R_2}}\,, \qquad \eta = \cos\Theta \tag{5.11}$$

one now finds after some simplifications

$$G(z,z') = \frac{1}{8\pi^3 R_1 R_2^3} \frac{1}{\sqrt{1-\xi^2}} \xi^{1+\frac{R_1}{R_2}} \frac{(1+\sqrt{1-\xi^2})^{\frac{R_1}{R_2}} - (1-\sqrt{1-\xi^2})^{\frac{R_1}{R_2}}}{\Big[(1+\sqrt{1-\xi^2})^{\frac{R_1}{R_2}} - 2\xi^{\frac{R_1}{R_2}} \cos\Theta + (1-\sqrt{1-\xi^2})^{\frac{R_1}{R_2}}\Big]^2}\,. \tag{5.12}$$

For the conformally flat case $R_1 = R_2 = R$, where (5.9) becomes the mass generated by the Weyl invariant coupling to the background, the above expression simplifies to

$$G(z,z') = \frac{1}{4\pi^3 R^4} \frac{\xi^2}{(2-2\xi\cos\Theta)^2} = \frac{1}{4\pi^3} \frac{1}{(u+v+i\varepsilon(t,t'))^2}\,, \tag{5.13}$$

where we have restored the $i\varepsilon(t,t')$-prescription. This result exactly matches (2.8).

The way to perform the KK mode summation on generic $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ backgrounds is very similar to the one presented above. One finds a linear relation between l and Δ

$$\Delta_\pm = \pm\frac{R_1}{R_2} l + \frac{dR_2 \pm d' R_1}{2R_2} \tag{5.14}$$

at the $(d + d' + 2)$-dimensional mass value

$$M^2 = \frac{d'^2 R_1^2 - d^2 R_2^2}{4 R_1^2 R_2^2}\,. \tag{5.15}$$

This expression is a generalization of (5.9) and it reduces to (2.6) in the conformally flat case. For generic dimension the way of computing the propagator is very similar to the one presented for the $\mathrm{AdS}_3 \times \mathrm{S}^3$ background. However the steps (5.4) to express the hypergeometric function in the AdS propagator and (5.10) to compute the sum become more tedious. For dealing with the hypergeometric functions see the remarks in appendix B. The sum generalizes in the way, that higher derivatives and more terms enter the expression (5.10).

Next we discuss the mode summation in the conformally flat case $R_1 = R_2$ at the Weyl invariant mass value but for generic d and d'. In this case with the corresponding conformal dimensions

$$\Delta = \Delta_+ = l + \frac{d+d'}{2}\,, \tag{5.16}$$

using (2.14) and (B.7), the propagator is expressed as

$$\begin{aligned} G(z,z') = {} & \frac{\Gamma(\frac{d'}{2})}{4\pi}\left(\frac{\xi}{2\pi R^2}\right)^{\frac{d+d'}{2}} \sum_{l=0}^{\infty} \frac{\Gamma(l+\frac{d+d'}{2})}{\Gamma(l+\frac{d'}{2})}\left(\frac{\xi}{2}\right)^{l} \times \\ & \times F\left(\frac{l}{2}+\frac{d+d'}{4}, \frac{l}{2}+\frac{d+d'}{4}+\frac{1}{2}; l+\frac{d'}{2}+1;\xi^2\right) C_l^{(\frac{d'}{2})}\left(1-\frac{v}{2R^2}\right). \end{aligned} \tag{5.17}$$

This equality together with the solution (2.8) has lead us to formulate a sum rule for the above given functions at generic d and d'. The above series should exactly reproduce (2.8). In appendix B we give an independent direct proof of the sum rule.

Considering the mode summation one finds an interpretation of the asymptotic behaviour of (2.8) observed in subsection 2.1. The asymptotic regime $u \to \infty$ corresponds to $\xi \to 0$. As the contribution of the lth mode is proportional to $\xi^{\Delta_+} \sim \xi^l$, the conformal dimension of the zero mode determines the asymptotic behaviour.

Note also that the additional singularity of (2.8) at the total antipodal position $z = \tilde{z}'$ can be seen already in (5.1). Under antipodal reflection in AdS_{d+1} the pure AdS propagator fulfills $G_{\Delta_\pm}(x,\tilde{x}') = (-1)^{\Delta_\pm} G_{\Delta_\pm}(x,x')$. On the sphere the spherical harmonics at antipodal points are related via $Y^I(y) = (-1)^l Y^I(\tilde{y})$. Hence, in case that $\Delta_\pm$ is given by (5.16), replacing z' by the total antipodal point $\tilde{z}'$ leads to the same expression for the mode sum up to an l-independent phase factor.

One final remark to the choice of Δ_+. What happens if one performs the mode expansion with AdS propagators based on Δ_-? First in any case for high enough KK modes Δ_- violates the unitarity bound (2.19). But ignoring this condition from physics one can nevertheless study the mathematical issue of summing with Δ_-. The corresponding series is given by (5.10) after replacing q by q^{-1}. It is divergent since for real u the variable q in (5.11) obeys $|q| \leq 1$ (case $R_1 = R_2$). One can give meaning to the sum by the following procedure. q as a function of ξ has a cut between $\xi = \pm 1$. If $|q| \leq 1$ on the upper side of the cut, then $|q| \geq 1$ on the lower side. Hence it is natural to define the sum with Δ_- as the analytic continuation from the lower side. By this procedure we found both for $\mathrm{AdS}_3 \times \mathrm{S}^3$ and $\mathrm{AdS}_5 \times \mathrm{S}^5$ up to an overall factor -1 the same result as using Δ_+. The sign factor can be understood as a consequence of the continuation procedure.

6. The plane wave limit

The plane wave background arises as a certain Penrose limit of $\mathrm{AdS}_5 \times \mathrm{S}^5$. The scalar propagator in the plane wave has been constructed in [17]. In this section we study how this propagator in the massless case arises as a limit of our $\mathrm{AdS}_5 \times \mathrm{S}^5$ propagator (2.8).

This approach is in the spirit of [15], where one follows the limiting process instead of taking the limit before starting any computations. One finds a simple interpretation of certain functions of the coordinates introduced in [17].

As an additional consistency check we take the $R \to \infty$ limit of the differential equation (2.1) using (2.5) to obtain the equation on the plane wave background and find that it is fulfilled by the massive propagator given in [17].

Taking the aforementioned Penrose limit of $\mathrm{AdS}_5 \times \mathrm{S}^5$ means to focus into the neighbourhood of a certain null geodesic which runs along an equator of the sphere with velocity of light. The metric of $\mathrm{AdS}_5 \times \mathrm{S}^5$ in global coordinates[7]

$$\mathrm{d}s^2 = R^2\big(- \mathrm{d}t^2 \cosh^2\rho + \mathrm{d}\rho^2 + \sinh^2\rho\, \mathrm{d}\Omega_3^2 + \mathrm{d}\psi^2 \cos^2\vartheta + \mathrm{d}\vartheta^2 + \sin^2\vartheta\, \mathrm{d}\hat{\Omega}_3^2\big) \tag{6.1}$$

via the replacements

$$t = z^+ + \frac{z^-}{R^2}\,, \qquad \psi = z^+ - \frac{z^-}{R^2}\,, \qquad \rho = \frac{r}{R}\,, \qquad \vartheta = \frac{y}{R} \tag{6.2}$$

in the $R \to \infty$ limit turns into the plane wave metric.

The relation between global coordinates and the embedding space coordinates, see e. g. [20], yields (ω_i, $\hat{\omega}_i$, $i = 1, \ldots, 4$ with $\vec{\omega}^2 = \vec{\hat{\omega}}^2 = 1$ are the embedding coordinates of the two unit 3-spheres)

$$u = 2R^2\Big[- 1 + \cosh\rho\cosh\rho'\cos(t-t') - \sinh\rho\sinh\rho'\,\omega_i\omega_i'\Big] \tag{6.3}$$

$$v = 2R^2\Big[+ 1 - \cos\vartheta\cos\vartheta'\cos(\psi-\psi') - \sin\vartheta\sin\vartheta'\,\hat{\omega}_i\hat{\omega}_i'\Big]\,. \tag{6.4}$$

Applying (6.2) one gets at large R up to terms vanishing for $R \to \infty$

$$\begin{aligned} u &= 2R^2\bigg[- 1 + \cos\Delta z^+ + \frac{1}{R^2}\bigg(- (\vec{x}^2 + \vec{x}'^2)\sin^2\frac{\Delta z^+}{2} + \frac{(\vec{x}-\vec{x}')^2}{2} - \Delta z^-\sin\Delta z^+\bigg)\bigg] \\ v &= 2R^2\bigg[+ 1 - \cos\Delta z^+ + \frac{1}{R^2}\bigg(- (\vec{y}^2 + \vec{y}'^2)\sin^2\frac{\Delta z^+}{2} + \frac{(\vec{y}-\vec{y}')^2}{2} - \Delta z^-\sin\Delta z^+\bigg)\bigg]\,, \end{aligned} \tag{6.5}$$

[7] The AdS_5 coordinates are related to the ones in (4.1) via $\cosh\rho = \sec\bar{\rho}$.

where $\Delta z^{\pm} = z^{\pm} - z'^{\pm}$ and $\vec{x} = r\vec{\omega}$, $\vec{y} = y\vec{\tilde{\omega}}$. In the $R \to \infty$ limit the sum of both chordal distances is thus given by

$$\Phi = \lim_{R\to\infty} (u + v) = -2(\vec{z}^2 + \vec{z}'^2)\sin^2\frac{\Delta z^+}{2} + (\vec{z} - \vec{z}')^2 - 4\Delta z^- \sin\Delta z^+ \,, \qquad (6.6)$$

where $\vec{z} = (\vec{x}, \vec{y})$, $\vec{z}' = (\vec{x}', \vec{y}')$ and Φ refers to the notation of [17]. Φ is precisely the $R \to \infty$ limit of the total chordal distance on $\mathrm{AdS}_5 \times \mathrm{S}^5$, which remains finite as both $\sim R^2$ terms in (6.5) cancel. This happens due to the expansion around a *null* geodesic. In appendix C it is shown that Φ is the chordal distance in the plane wave.

The massless propagator in the plane wave background in the $R \to \infty$ limit of (2.8) with $d = d' = 4$ thus becomes

$$G_{\mathrm{pw}}(z, z') = \frac{3}{2\pi^5}\frac{1}{(\Phi + i\varepsilon(z^+, z'^+))^4} \,, \qquad (6.7)$$

which agrees with [17].

In addition we checked the massive propagator of [17] which fulfills the differential equation on the plane wave background. This equation can be obtained from (2.1) and (2.5) by taking the $R \to \infty$ limit. In the limit the sum of both chordal distances is given in (6.6). The difference is given by

$$\lim_{R\to\infty} \frac{u - v}{R^2} = 4(\cos\Delta z^+ - 1) \qquad (6.8)$$

this has to be substituted into (2.5). Finally, one obtains the differential equation

$$\left[4\cos\Delta z^+\left(5\frac{\partial}{\partial\Phi} + \Phi\frac{\partial^2}{\partial\Phi^2}\right) + 4\sin\Delta z^+\frac{\partial}{\partial\Phi}\frac{\partial}{\partial\Delta z^+} - M^2\right]G_{\mathrm{pw}}(z, z') = \frac{i}{\sqrt{-g_{\mathrm{pw}}}}\delta(z, z') \,, \qquad (6.9)$$

which is fulfilled by the expression given in [17]. As already noticed in section 2, in contrast to the massless propagator the massive one depends not only on the total chordal distance Φ but in addition on (6.8).

7. Conclusions

In this paper we have focussed on the propagator of scalar fields on $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ backgrounds. We have argued that for an investigation of holography in the plane wave, in a first step one should study this propagator instead of the bulk-to-boundary one, since only for the former the Penrose limit is well defined.

We have first discussed solutions of the defining propagator equation at points away from possible singularities. On conformally flat backgrounds and for Weyl invariant coupled fields, both in $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ and in pure AdS_{d+1}, exist two solutions, which are powerlike in the chordal distances. For $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ these two solutions are powers either in the sum or difference of the chordal distances with respect to the AdS_{d+1} and $\mathrm{S}^{d'+1}$ factor. The first solution has a singularity if both points coincide or if they are at antipodal positions to each other. The second solution has singularities where both points are semi-antipodal to each other.

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Whether, acting with the d'Alembert operator, δ-sources are generated at the locations of these singularities, depends on the time ordering prescription. For AdS_{d+1}, being the universal cover of the embedded hyperboloid, standard time ordering is appropriate. Then for both solutions source terms arise only at coinciding times. This implies that the first solution develops just the right source to solve the full propagator equation. But the second solution necessarily has a source term away from the coincidence of the two points. Therefore it cannot be used to form different propagators via linear combinations with the first solution.

This is in contrast to the pure AdS_{d+1} case. There the second solution has a singularity at the position where both points are antipodal to each other on the hyperboloid, i.e. there is no singularity at coinciding time coordinates. Hence, this singularity does not lead to a δ-source. Thus, taking the sum and difference of the two solutions, propagators, which obey respectively Neumann and Dirichlet boundary conditions, can be constructed.

In addition, for $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ we have investigated the KK decomposition of the propagator using spherical harmonics. We have noted that the summation can be performed even in non conformally flat backgrounds, but only for special mass values. The relevant condition is that the conformal dimension of the field mode is a linear function of the KK mode parameter.

In the conformally flat case for a Weyl invariant coupled field the uniqueness of the solution of the differential equation in combination with the KK decomposition led to the formulation of a theorem that sums up a product of Legendre functions and Gegenbauer polynomials. It was independently proven in appendix B.

For $\mathrm{AdS}_5 \times \mathrm{S}^5$ we explicitly performed the Penrose limit on our expression for the propagator to find the result on the plane wave background. We found agreement with the literature [17] and got an interpretation for the spacetime dependence of the result. It simply depends on the $R \to \infty$ limit of the sum of both chordal distances on $\mathrm{AdS}_5 \times \mathrm{S}^5$, which was shown to be the chordal distance in the plane wave. In the general massive case there is an additional dependence on the suitable rescaled difference of both chordal distances. We formulated the differential equation in the limit and checked that the massive propagator on the plane wave background given in [17] is a solution.

Clearly future work is necessary to construct the propagator for the case of generic mass values. But already with our results one should be able to address the issue of defining a bulk-to-boundary propagator in the plane wave limit.

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A. Relation of the bulk-to-bulk and the bulk-to-boundary propagator

We will show for a scalar field in an euclidean space how the bulk-to-boundary propagator is related to the bulk-to-bulk propagator, if the boundary has codimension one w.r.t. the bulk, like in the case of the AdS/CFT correspondence. The bulk-to-bulk propagator $G(x,x')$ of a scalar field with mass m is defined as Green function that fulfills[8]

$$(\Box_x - m^2)G(x,x') = -\frac{1}{\sqrt{g}}\delta(x,x'), \tag{A.1}$$

with appropriate boundary conditions. Here $\Box_x$ is the Laplace operator on the $(d+1)$-dimensional riemannian manifold M with a d-dimensional boundary which we denote with ∂M. The coordinates are x^i, the metric is g_{ij} and its determinant is g. The propagator $G(x,x')$ corresponds to a scalar field $\phi(x)$ which should obey

$$(\Box_x - m^2)\phi(x) = J(x), \qquad \lim_{x_\perp \to 0} \phi(x) x_\perp^a = \bar{\phi}(\bar{x}), \tag{A.2}$$

where $J(x)$ are sources for the field ϕ in the interior. We have split the coordinates like $x = (x_\perp, \bar{x})$ with the boundary at $x_\perp = 0$. Boundary values $\bar{\phi}$ for the field ϕ are specified with a nontrivial scaling with $x_\perp^a$ for later convenience. The bulk-to-boundary propagator $K(x,\bar{x}')$ is defined as the solution of the equations

$$(\Box_x - m^2)K(x,\bar{x}') = 0, \qquad \lim_{x_\perp \to 0} K(x,\bar{x}') x_\perp^a = \delta(\bar{x},\bar{x}'), \tag{A.3}$$

where the second equation implements the necessary singular behaviour at the boundary. A solution of the equations (A.2) with $J(x) = 0$ is then given by

$$\phi(x) = \int_{\partial M} \mathrm{d}^d\bar{x}' K(x,\bar{x}')\bar{\phi}(\bar{x}'). \tag{A.4}$$

Since we deal with the problem in euclidean signature [29, 30, 3], we will denote $K(x,\bar{x}')$ as the Poisson kernel. It is not independent from the Green function defined via (A.1) as we will now show.

With (A.1) one can write an identity for the field ϕ that reads

$$\phi(x) = -\int_M \mathrm{d}^{d+1}x' \sqrt{g}\phi(x')(\Box_{x'} - m^2)G(x,x'). \tag{A.5}$$

After applying partial integration twice and using (A.2) it assumes the form

$$\phi(x) = \int_{\partial M} \mathrm{d}A'_\mu \sqrt{g} g^{\mu\nu}\left[(\partial'_\nu \phi(x'))G(x,x') - \phi(x')\partial'_\nu G(x,x')\right] - \int_M \mathrm{d}^{d+1}x' \sqrt{g} J(x') G(x,x'), \tag{A.6}$$

where $\mathrm{d}A'_\mu$ denotes the infinitesimal area element on ∂M which points into the outer normal direction and ∂'_μ denotes a derivative w.r.t. x'_μ. If one has the additional restriction that $G(x,x') = 0$ for $x' \in \partial M$ ($x'_\perp = 0$) the first term in the above boundary integral is zero.

[8] This equation is the analytic continuation of (2.1).

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One then arrives at the 'magic rule' which for the boundary value problem in presence of a source $J(x)$ in the interior can be found in [31]. Here, however, we have to be more careful. In (A.2) we have allowed for a scaling of the boundary value with $x_\perp^a$ as written down in (A.3). For $a > 0$ the vanishing $G(x, x')$ at the boundary can be compensated and the first term in the boundary integral of (A.6) then contributes.

Considering AdS_{d+1}, this is indeed the case, because the field ϕ with conformal dimension Δ represents the non-normalizable modes, which scale as given in (A.3) with $a = \Delta - d$. Indicating the corresponding propagator with the suffix Δ, one finds $G_\Delta(x, x') = 0$ at $x' \in \partial M$ but the vanishing is compensated by the singular behaviour of the non-normalizable modes in the limit $x'_\perp \to 0$.

We now formulate (A.6) on euclidean AdS_{d+1} in Poincaré coordinates with $J(x') = 0$, where one has

$$\mathrm{d}A_\mu = -\,\mathrm{d}^d\bar{x}\delta_\mu^\perp\,, \qquad \sqrt{g} = \left(\frac{R_1}{x_\perp}\right)^{d+1}, \qquad g^{\perp\perp} = \left(\frac{x_\perp}{R_1}\right)^2. \tag{A.7}$$

The minus sign in the area element stems from the fact that the $x_\perp$-direction points into the interior of AdS_{d+1}, but one has to take the outer normal vector. Now (A.6) reads

$$\phi_\Delta(x) = -R_1^{d-1}\int \mathrm{d}^d\bar{x}' x'^{1-d}_\perp\left[(\partial'_\perp\phi(x'))G_\Delta(x,x') - \phi(x')\partial'_\perp G_\Delta(x,x')\right]. \tag{A.8}$$

Using now (A.3) with $a = \Delta - d$, one finds that the relation of the bulk-to-bulk and the bulk-to-boundary propagator is given by

$$K_\Delta(x, \bar{x}') = -R_1^{d-1}\left[(d-\Delta)x'^{-\Delta}_\perp - x'^{1-\Delta}_\perp\partial'_\perp\right]G_\Delta(x,x')\big|_{x'_\perp=0}\,. \tag{A.9}$$

If we now insert the explicit expression (2.14) for $G_\Delta(x, x')$, we see what we already mentioned: in approaching the boundary ($x'_\perp \to 0$), $G_\Delta(x, x')$ itself goes to zero like $x'^\Delta_\perp$ but this is compensated by the singular behaviour of the prefactor in the first term of (A.9). Hence, in contrast to the situation of the 'magic rule' [31], it contributes to the bulk-to-boundary propagator. One then finds with

$$\xi = \frac{2x_\perp x'_\perp}{x_\perp^2 + x'^2_\perp + (\bar{x} - \bar{x}')^2}\,. \tag{A.10}$$

in Poincaré coordinates and with $F(a, b; c; 0) = 1$ that effectively

$$K_\Delta(x, \bar{x}') = -R_1^{d-1}(d-2\Delta)x'^{-\Delta}_\perp G_\Delta(x,x')\big|_{x'_\perp=0}\,, \tag{A.11}$$

which is in perfect agreement with the explicit expressions given in [3].

B. Useful relations for hypergeometric functions and spherical harmonics

Most of the relations we present here can be found in [32, 33, 34] or are derived from there. The hypergeometric functions in the propagators (2.14) with $\Delta_\pm = \frac{d\pm1}{2}$ (at the mass value generated by the Weyl invariant coupling) become ordinary analytic expressions

$$
\begin{aligned}
F\left(a, a+\frac{1}{2}; \frac{1}{2}; \xi^2\right) &= \frac{1}{2}\left[(1+\xi)^{-2a} + (1-\xi)^{2a}\right], \\
F\left(a+\frac{1}{2}, a+1; \frac{3}{2}; \xi^2\right) &= -\frac{1}{4a\xi}\left[(1+\xi)^{-2a} - (1-\xi)^{-2a}\right].
\end{aligned} \tag{B.1}
$$

Setting $a = \frac{d-1}{4}$ one finds (2.17).

To find the hypergeometric functions relevant for the propagators in higher dimensional AdS spaces one can use a recurrence relation (Gauß' relation for contiguous functions)

$$
\begin{aligned}
F(a,b;c-1;z) &= \frac{c[c-1-(2c-a-b-1)z]}{c(c-1)(1-z)} F(a,b;c;z) + \\
&\quad + \frac{(c-a)(c-b)z}{c(c-1)(1-z)} F(a,b;c+1;z) \ .
\end{aligned} \tag{B.2}
$$

where the hypergeometric functions relevant in lower dimensional AdS spaces enter.

For odd AdS dimensions (even d) the relevant hypergeometric functions can be expressed with the above recurrence relation in terms of ordinary analytic functions. This happens because of the explicit expressions

$$
\begin{aligned}
F\left(a, a+\frac{1}{2}; 2a; z\right) &= \frac{2^{2a-1}}{\sqrt{1-z}}\left[1+\sqrt{1-z}\right]^{1-2a}, \\
F\left(a, a+\frac{1}{2}; 2a+1; z\right) &= 2^{2a}\left[1+\sqrt{1-z}\right]^{-2a}.
\end{aligned} \tag{B.3}
$$

One has to apply (B.2) n times to compute the AdS propagator at generic Δ in $d+1 = 3+2n$ dimensions. For AdS_3 one simply uses the first expression in (B.3). The AdS_5 case is of particular importance and therefore we give the explicit expression for the needed hypergeometric function

$$
F\left(\frac{\Delta}{2}, \frac{\Delta}{2}+\frac{1}{2}; \Delta-1; z\right) = \frac{1}{2(1-z)^{\frac{3}{2}}}\left[\frac{2}{1+\sqrt{1-z}}\right]^{\Delta-1}\left[\sqrt{1-z}+\frac{\Delta-1}{\Delta-2}(1-z)+\frac{1}{\Delta-1}\right]. \tag{B.4}
$$

Spherical harmonics $Y^I(y)$ on $S^{d'+1}$ are characterized by quantum numbers

$$
I = (l, m_1, \dots, m_{d'}), \qquad l \geq m_1 \geq \dots \geq m_{d'-1} \geq |m_{d'}| \geq 0 \tag{B.5}
$$

and form irreducible representations of $SO(d'+2)$. They are eigenfunctions with respect to the Laplace operator on the sphere

$$
\Box_y Y^I(y) = -\frac{l(l+d')}{R_2^2} Y^I(y) \tag{B.6}
$$

and satisfy the relation

$$
\begin{aligned}
\sum_{m_1 \geq \dots \geq m_{d'-1} \geq |m_{d'}| \geq 0}^{l} Y^I(y) Y^{*I}(y') &= \frac{(2l+d')\Gamma(\frac{d'}{2})}{4\pi^{\frac{d'}{2}+1}} C_l^{(\frac{d'}{2})}(\cos\Theta), \\
\cos\Theta = \frac{\vec{Y}\cdot\vec{Y}'}{R_2^2} &= 1 - \frac{v}{2R_2^2}.
\end{aligned} \tag{B.7}
$$

The $C_l^{(\frac{d'}{2})}$ are the Gegenbauer Polynomials which can be defined via their generating function

$$\frac{1}{(1-2q\eta+q^2)^\beta} = \sum_{l=0}^{\infty} q^l C_l^{(\beta)}(\eta)\,. \tag{B.8}$$

Using (2.14) and (B.7) for $\Delta_+ = l + \frac{d+d'}{2}$ (leading to (5.17)), one finds the solution (2.8) if the following relation holds for $\alpha \geq \beta > 0$, $2\alpha, 2\beta \in \mathbb{N}$

$$\sum_{l=0}^{\infty} \frac{\Gamma(l+\alpha)}{\Gamma(l+\beta)} \left(\frac{\xi}{2}\right)^l F\left(\frac{l}{2}+\frac{\alpha}{2}, \frac{l}{2}+\frac{\alpha}{2}+\frac{1}{2}; l+\beta+1; \xi^2\right) C_l^{(\beta)}(\eta) = \frac{\Gamma(\alpha)}{\Gamma(\beta)} \frac{1}{(1-\xi\eta)^\alpha}\,, \tag{B.9}$$

with the interpretation $\alpha = \frac{d+d'}{2}$, $\beta = \frac{d'}{2}$. We could not find the above formula in the literature. It is in fact a summation rule for a product of a special hypergeometric function for which so called quadratic transformation formulae exist and which can be expressed in terms of a Legendre function [33] and of a Gegenbauer polynomial. The identity can therefore be re-expressed in the following way

$$\left(\frac{2}{\xi}\right)^\beta (1-\xi^2)^{\frac{\beta-\alpha}{2}} \sum_{l=0}^{\infty} \Gamma(l+\alpha)(l+\beta) P_{\alpha-\beta-1}^{-l-\beta}\left(\frac{1}{\sqrt{1-\xi^2}}\right) C_l^{(\beta)}(\eta) = \frac{\Gamma(\alpha)}{\Gamma(\beta)} \frac{1}{(1-\xi\eta)^\alpha}\,. \tag{B.10}$$

The simplest way to prove[9] is to use the orthogonality of the Gegenbauer polynomials

$$\int_{-1}^{1} d\eta\ (1-\eta^2)^{\beta-\frac{1}{2}} C_m^{(\beta)}(\eta) C_n^{(\beta)}(\eta) = \frac{2^{1-2\beta}\pi\Gamma(n+2\beta)}{\Gamma(n+1)(n+\beta)\Gamma(\beta)^2}\delta_{mn} \tag{B.11}$$

to project out a term with fixed l from the sum in (B.9). The Gegenbauer polynomials in the integral on the r.h.s. of (B.9) should then be expressed via Rodrigues' formula

$$C_n^{(\beta)}(\eta) = (-1)^n 2^{-n} \frac{\Gamma(\beta+\frac{1}{2})\Gamma(n+2\beta)}{\Gamma(n+1)\Gamma(2\beta)\Gamma(n+\beta+1)} (1-\eta^2)^{\frac{1}{2}-\beta} \frac{d^n}{d\eta^n}(1-\eta^2)^{n+\beta-\frac{1}{2}}\,. \tag{B.12}$$

Repeated partial integration to shift all the above derivatives to the first factor under the integral and a suitable variable transformation at the end leads to an integral that can be expressed via a hypergeometric function. This hypergeometric function is than connected to the one on the left hand side of (B.9) via the quadratic transformation formula

$$F\left(a, a+\frac{1}{2}; c; z^2\right) = (1+z)^{-2a} F\left(2a, c-\frac{1}{2}; 2c-1; \frac{2z}{1+z}\right)\,. \tag{B.13}$$

Both sides of (B.9) than match and the proof is complete. The relation (B.9) is therefore valid not only for $\alpha \geq \beta > 0$, $2\alpha, 2\beta \in \mathbb{N}$ but for all $\beta > 0$.

[9] We thank Danilo Diaz for delivering a simplification of our proof in the previous version, that allows for an extension to more general values of α and β.

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C. The chordal distance in the plane wave

It has been shown by Penrose [35] that it is impossible to globally embed the plane wave spacetimes into a pseudo-euclidean spacetime. However, an isometric embedding of the D-dimensional CW spaces with metric

$$\mathrm{d}s^2 = -4\,\mathrm{d}z^+\,\mathrm{d}z^- + H_{ij}z^iz^j(\mathrm{d}z^+)^2 + \mathrm{d}\vec{z}^2\,. \tag{C.1}$$

in $\mathbb{R}^{2,D}$ is possible [8]. The flat metric of $\mathbb{R}^{2,D}$ via the coordinate transformations

$$\begin{aligned} Z_+^1 &= \frac{1}{2}(Z_0 + Z_d)\,, & Z_-^1 &= \frac{1}{2}(Z_0 - Z_d)\,, \\ Z_+^2 &= \frac{1}{2}(Z_{d+1} + Z_{d-1})\,, & Z_-^2 &= \frac{1}{2}(Z_{d+1} - Z_{d-1}) \end{aligned} \tag{C.2}$$

can be transformed to

$$\mathrm{d}s^2 = -4\sum_{k=1}^{2}\mathrm{d}Z_+^k\,\mathrm{d}Z_-^k + \sum_{i=1}^{D}\mathrm{d}Z_i^2\,. \tag{C.3}$$

If the hypersurface is defined as

$$\sum_{k=1}^{2} Z_+^kZ_+^k = 1\,, \qquad H_{ij}Z^iZ^j + 4\sum_{k=1}^{2}Z_+^kZ_-^k = 0 \tag{C.4}$$

and parameterized as follows

$$\begin{aligned} Z_+^1 &= \cos z^+\,, \\ Z_-^1 &= -z^-\sin z^+ - \frac{1}{4}H_{ij}z^iz^j\cos z^+\,, \\ Z_+^2 &= \sin z^+\,, \\ Z_-^2 &= z^-\cos z^+ - \frac{1}{4}H_{ij}z^iz^j\sin z^+\,, \\ Z_i &= z_i\,, \end{aligned} \tag{C.5}$$

one finds that the induced metric is given by (C.1). The chordal distance in the plane wave reads

$$\begin{aligned} \Phi(z,z') &= -4\sum_{k=1}^{2}(Z_+^k(z) - Z_+^k(z'))(Z_-^k(z) - Z_-^k(z')) + \sum_{i=1}^{D}(Z_i(z) - Z_i(z'))^2 \\ &= -4(z^- - z'^-)\sin(z^+ - z'^+) + 2H_{ij}(z^iz^j + z'^iz'^j)\sin^2\frac{z^+ - z'^+}{2} + (\vec{z} - \vec{z}')^2\,. \end{aligned} \tag{C.6}$$

In our case where $H_{ij} = -\delta_{ij}$, this result matches (6.6).

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